NSF GK-12 MAVS Project Lesson Plan Template

Lesson 4: SOLUTIONS TO SYSTEMS OF LINEAR INEQUALITIES IN PHASE PLANE ANALYSIS

GK-12 MAVS Fellow: Alice Lubbe **GK-12 MAVS Mentor Teacher: Alicia Geppert**

Class: Algebra 1

Topic: Graphing Systems of Linear Inequalities & Phase Plane Analysis

Objectives: Curriculum objective: Students will be able to investigate the solutions to a system of inequalities by graphing the inequality and shading appropriately. Research objectives: Students will explore and discover the significance of the solution region of a system of linear inequalities as applicable to slope fields associated with a system of differential equations. Students will explore and discover the relationship between the phase plane analysis of two variables and the associated solution curves of those variables over time.

Standards: TEKS - (7) Linear functions. The student formulates equations and inequalities based on linear functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation. The student is expected to: (B) investigate methods for solving linear equations and inequalities using concrete models, graphs, and the properties of equality, select a method, and solve the equations and inequalities.

NCTM- Represent and analyze mathematical situations and structures using algebraic symbols - write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency—mentally or with paper and pencil in simple cases and using technology in all cases;

Key vocabulary: slope, y-intercept, system, inequality, solution, equilibrium, phase plane, trajectory, nullcline, periodic, stable

Materials and Resources: Floor graph: an open 12 ft. X 12 ft. space in the middle of the room; the x and y axes consist of "Duck" tape with axes marking a [– 6 , 6] X [– 6 , 6] grid. Set of 10 arrows, each 2 in. wide, of following lengths: 4 long (30 in.), 2 medium (16 in.), and 4 short (9 in.). Arrows are laminated and affixed with velcro buttons so they will stick on carpeted floor. Biological differential equation software called XPPAUT was used to generate graphs. Powerpoint files were displayed as well as documents via document camera.

Research Setting/Connection/Motivation: The potential difference (mV) of neuron cell membranes is typically modeled by a nonlinear system of ordinary differential equations. Phase plane analysis is instrumental in characterizing the types of solutions that can be generated, depending upon the values of certain parameters and initial conditions. In order to demonstrate this analysis, a simpler linear system in two variables (x and y) can be used with Algebra 1 students.

Prior knowledge: The students have graphed lines, and graphed systems of equations.

Lesson Presentation:

A) Lesson 4 Intro Slides 1 – 9 (Research background, **Engage**, 10 min)

B) EXPLORE SYSTEMS OF INEQUALITIES activity (pdf)

1. Problem #1, warm-up (**Explore**, 10 min): After students complete, show solution on document camera.
2. Problem #2 and brief notes (**Explore & Explain**, 15 min): After students complete #2, show solution on document camera. Show notes on document camera.
3. MORE PRACTICE (**Extend**, 10 min)

C) PHASE PLANE ANALYSIS IS A "CAKE WALK", a "groups-of-three" activity (pdf)

1. STEP ONE (**Explore & Evaluate**, 10 min): Piece of "cake" (piece of candy like starburst or fun-size skittles) provided upon correct completion.
2. STEP TWO (**Extend & Evaluate**, 15 min): Demonstrate via example, as shown in table. Allow student volunteers to place trajectory arrow for this example on a floor graph. Piece of "cake" provided upon correct completion.
3. STEP THREE (**Engage & Evaluate**, 5 min): Piece of "cake" provided upon correct completion.

Each group will receive one or two arrows (laminated cut-outs with velcro). The arrows represent trajectories in *x* and *y* for the phase plane analysis of a system of two linear differential equations, where *x* and *y* are functions of time. The base of each arrow will be placed on an initial point on the floor graph. (Each initial point given to a group will lie in the solution region for that group.) Students must compute the direction of the trajectory in order to properly point the arrow on the floor graph. Arrows in any given region point in the same direction.

DIRECTION KEY: UP-LEFT / DOWN-LEFT / DOWN-RIGHT / UP-RIGHT

4. OBSERVATIONS & NOTES (**Engage & Explain**, 5 – 10 min)

Students can be selected to stand and walk around the phase plane trajectory "oval", as other students assess changes in the variables x and y. Since these variables change over time, the connection of the phase plane analysis to the graphs of the solution of the system of differential equations can be inferred here. In the research wrap-up, graphs of these solutions are shown.

D) Lesson 4 Intro Slides 10 – 12 (Research wrap-up, **Explain**, 5 – 10 min)

TRAJECTORY KEY & GROUP ASSIGNMENTS

SYSTEM	INITIAL POINT	FINAL POINT*	ARROW LENGTH	# OF GROUPS
#1 $\begin{cases}2x-5y<0\\3x-2y>0\end{cases}$	(4 , 2) (3 , 3)	(2 , 10) (− 6 , 6)	Short Medium	1 or 2
#2 $\begin{cases}2x-5y<0\\3x-2y<0\end{cases}$	(1 , 3) (− 1 , 2.2) (− 4 , − 1)	(− 12 , 0) (− 14 , − 5.2) (− 7 , − 11)	**Long** **Long** Short	2
#3 $\begin{cases}2x-5y>0\\3x-2y<0\end{cases}$	(− 4 , − 2) (− 3 , − 3)	(− 2 , − 10) (6 , − 6)	Short Medium	1 or 2
#4 $\begin{cases}2x-5y>0\\3x-2y>0\end{cases}$	(− 1 , − 3) (1 , − 2.2) (4 , 1)	(12 , 0) (14 , 5.2) (7 , 11)	**Long** **Long** Short	2

*** FINAL POINT** is the calculated ordered pair toward which the associated arrow points.
The group with the **Long** arrows can be groups with stronger math skills.

Vertical Strands:

6th Grade: Local: use pictorial models and input/output tables to represent equations.

7th Grade: 7.4C use words and symbols to describe the relationship between the terms in an arithmetic sequence (with a constant rate of change) and their positions in a sequence.

8th Grade: 8.5B find and evaluate an algebraic expression to determine any term in an arithmetic sequence with a constant rate of change.

Algebra I - A.7B investigate methods for solving linear equations and inequalities using concrete models, graphs, and the properties of equality, select a method, and solve the equations and inequalities.

Algebra II – 2A.3A analyze situations and formulate systems of equations in two or more unknowns or inequalities in two unknowns or inequalities in two unknowns to solve problems.

Fundamentals of Algebra (M0302): The students will be able to simplify and solve linear and quadratic equations, systems of equations and inequalities.

College Algebra (M1302): Students will be able to solve algebraic equations and simultaneous systems of equations. They will also be able to interpret the meaning of the solution(s) and demonstrate graphical solution techniques when appropriate.

Precalculus I (M1322): Students will be able to solve problems by applying elementary algebra: properties of the real number system and complex number system, using rules for exponents and radicals, simplifying algebraic expressions, solving equations and inequalities.

Mathematical Models (M3315): Students will be able to analyze the solutions from differential equation models and determine whether the solutions are consistent with measurement data and/or observations.

Differential Equations and Linear Algebra (M3319): Students will be able to solve systems of linear algebraic equations and systems of first-order linear differential equations.

Teacher Reflections:

All in all this was a great lesson for the students to learn how to solve systems of linear inequalities. Thus the curriculum objectives were met. As for the significance of the solution region of a system of linear inequalities as applicable to the slope fields associated with a system of differential equations, the students adequately explored and discovered this connection. The use of a large floor graph and various-sized arrows to represent solution trajectories in phase plane analysis was quite engaging for the students. We discovered after the first class, that when students placed their arrows on the floor graph, it worked best if one student stood on the initial point, while a second student stood on the coordinates toward which the arrow should point.

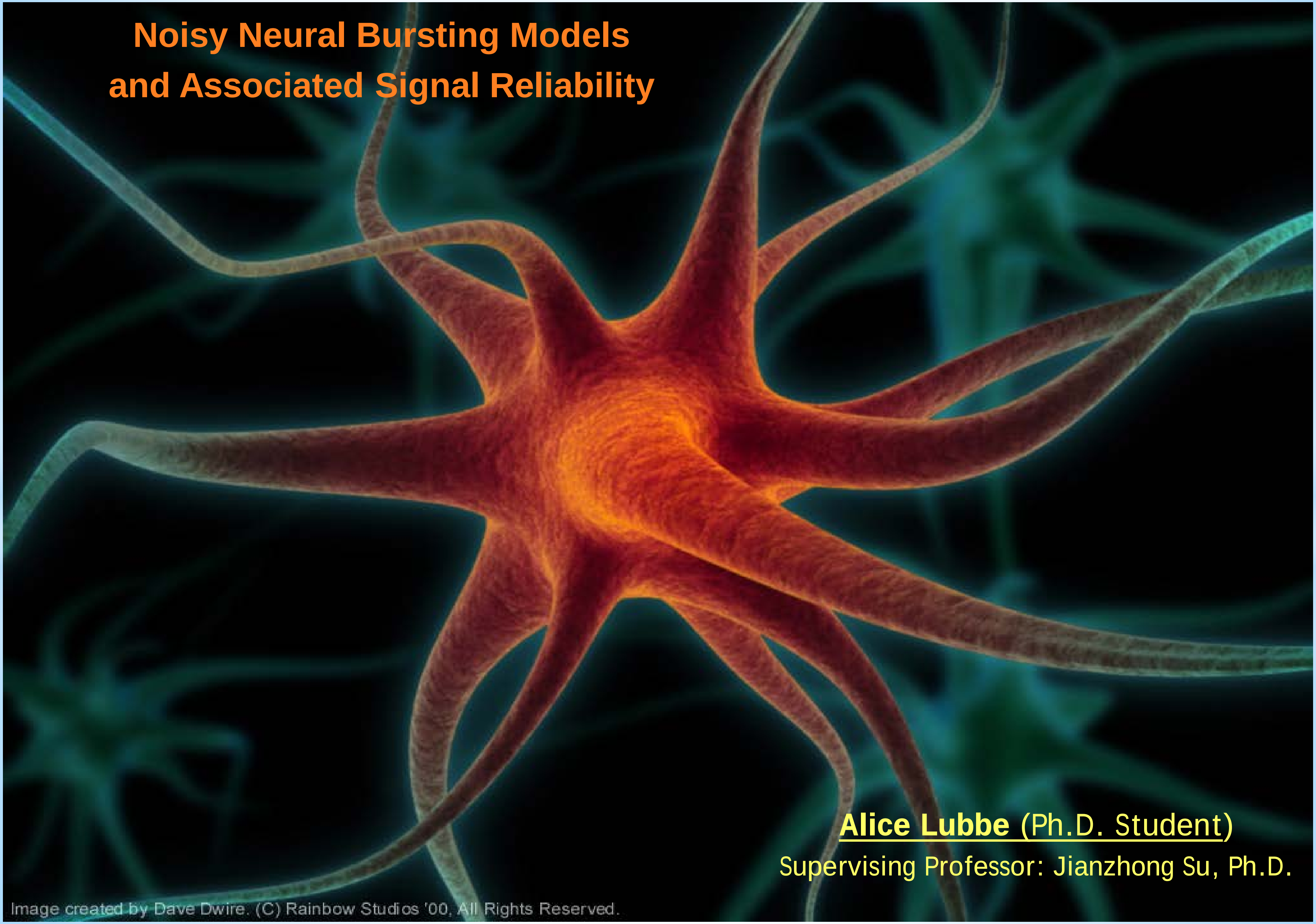
Time management is a key to executing this lesson. In most of the lessons we did, parts A) and B) took 50 – 55 minutes instead of the desired 45 minutes. And that was even with skipping #3 of part B) above. Only one group out of all of the classes had time to work on B), #3. About one-fourth of the groups completed STEP FOUR of the PHASE PLANE ANALYSIS IS A "CAKE WALK" activity. Also, in four of the six classes, the research wrap-up was limited to approximately 5 minutes due to going over time in previous sections. However it should be noted that the use of candy rewards for accurately completing the STEPS in the "CAKE WALK" activity motivated students to work harder and faster on the assignment.

During the research wrap-up portion at the end, a student in one class asked what the difference was between the graph of solutions $x(t)$ and $y(t)$ and the graph of the phase plane. It was explained to the student that the phase plane is simply a graph of the relationship between x (level of free calcium) and y (voltage level) without considering the effect of *time*. (*Note*: The use of a simple linear system of differential equations to model the behavior of neural cell membranes is oversimplified. This is explained to the students. However, the phase plane analysis is quite relevant to solving systems of ordinary differential equations.)

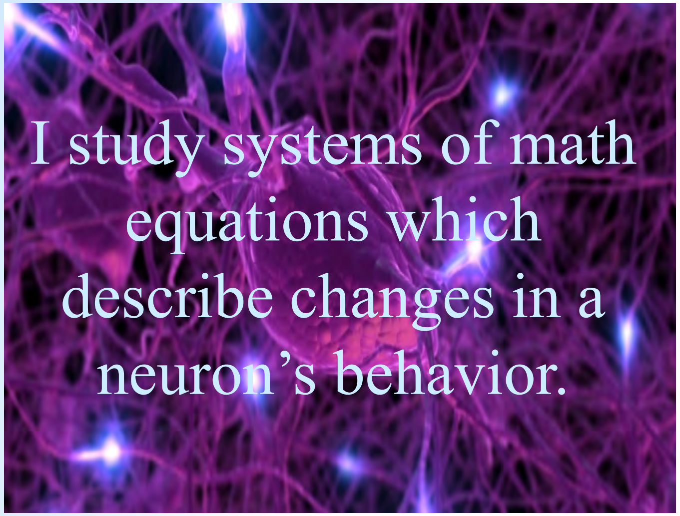
Alice Lubbe, Math Student

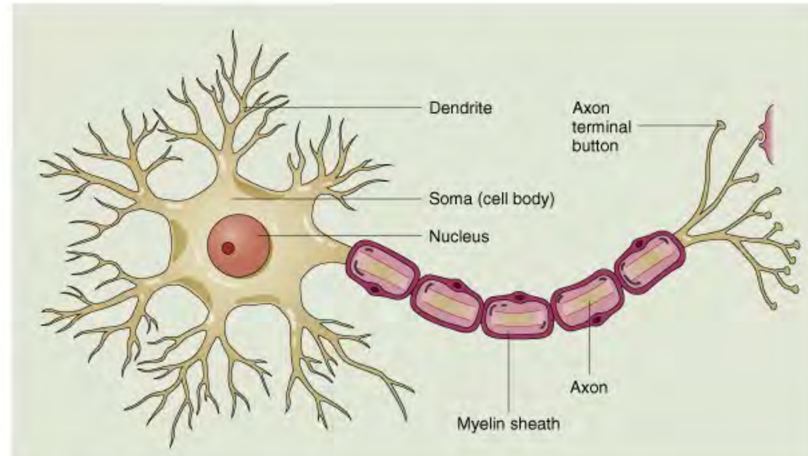
remember the bats →

Noisy Neural Bursting Models and Associated Signal Reliability

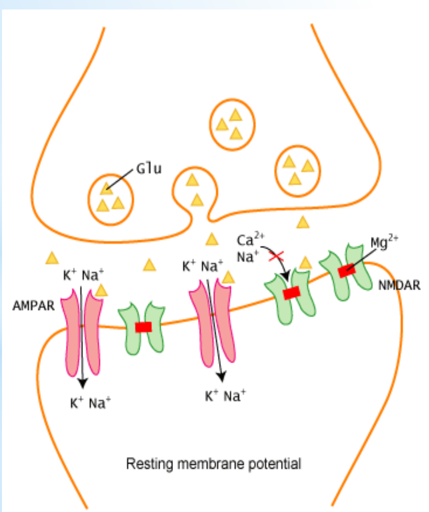


Alice Lubbe (Ph.D. Student)
Supervising Professor: Jianzhong Su, Ph.D.

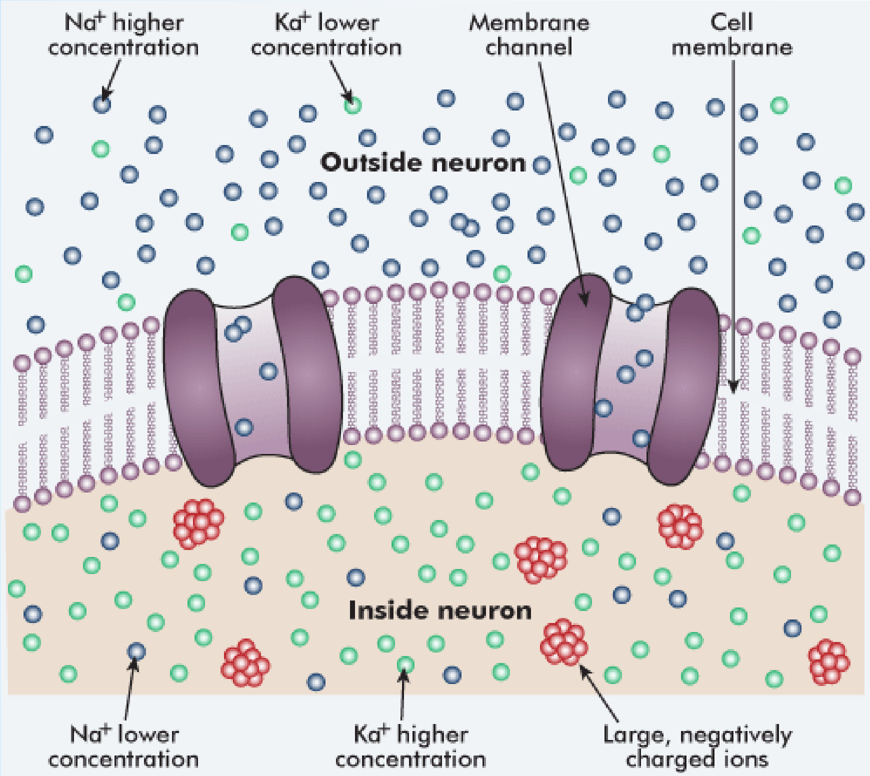




Neurons are "nerve cells" which are building blocks to the nervous system. Electrical pulses (signals) transmit information from the cell's nucleus, through the axon, to axon terminals . . .

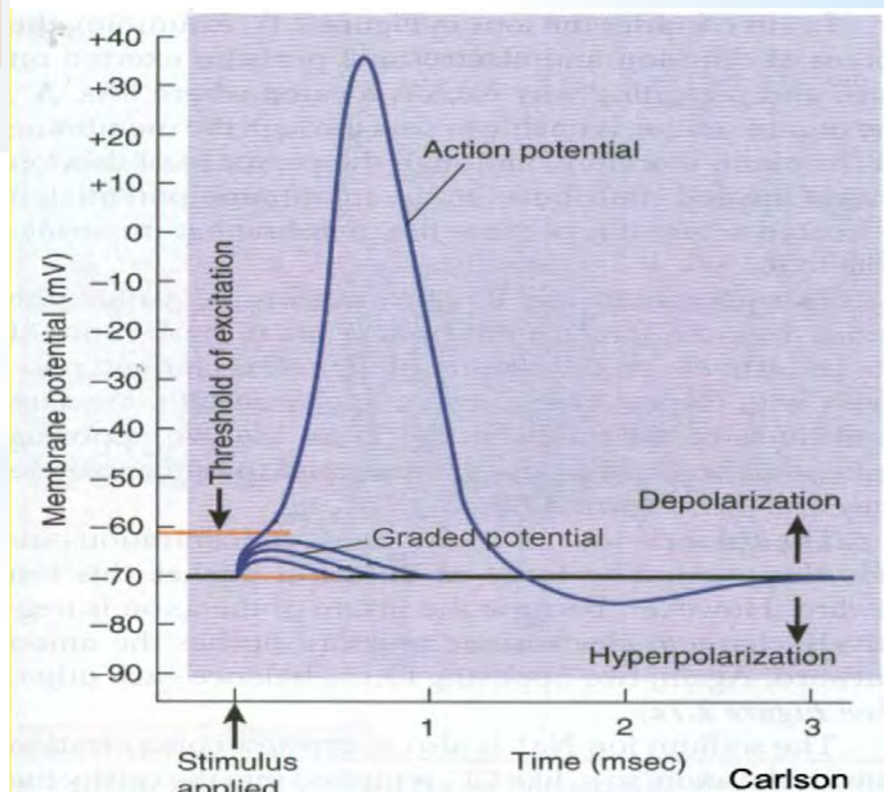


The electrical signals are converted into chemical signals, which are then transmitted via the synapse to another cell.



cell membrane

The cell membrane has channels through which ions flow. The membrane maintains a "concentration gradient", which can be opposed by a growing electrical potential that opens channels and allows ions to flow into or out of the cell.

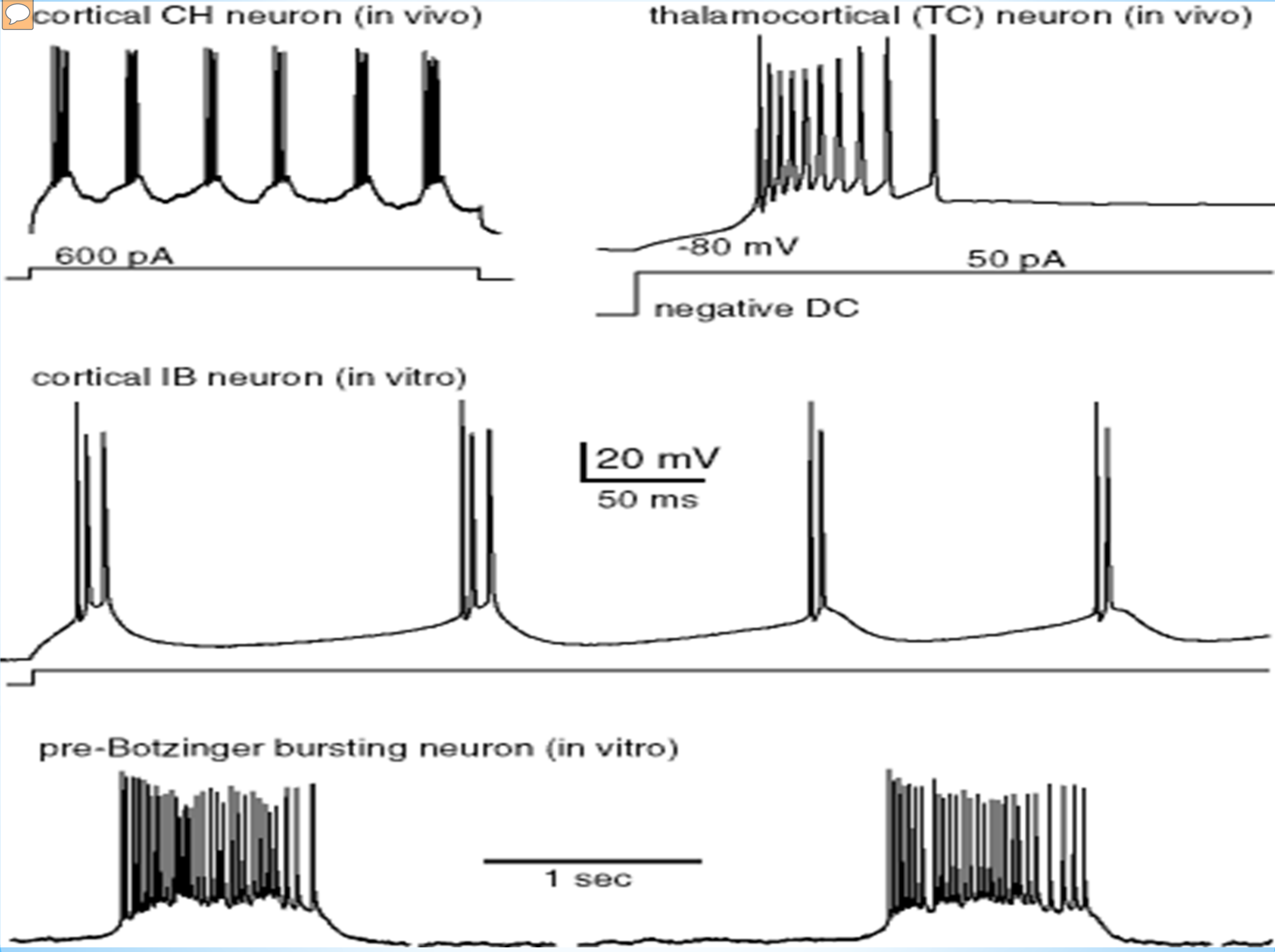


Meaningful pursuits:

- Improve treatment for epilepsy
- Help find cure for Parkinson's
- Understand drug addiction

Obtaining measurements:

- External electrodes
- Deep brain stimulation


cortical CH neuron (in vivo)
600 pA
thalamocortical (TC) neuron (in vivo)
-80 mV
50 pA
negative DC
cortical IB neuron (in vitro)
20 mV
50 ms
pre-Botzinger bursting neuron (in vitro)
1 sec

A General Bursting Model Example with Channel Noise

fast:

(1) $$\frac{du}{dt} = y$$

(2) $$\frac{dy}{dt} = -F(u)y - G(u) - z - \epsilon[h(u) - z] - \sigma\Gamma_t$$

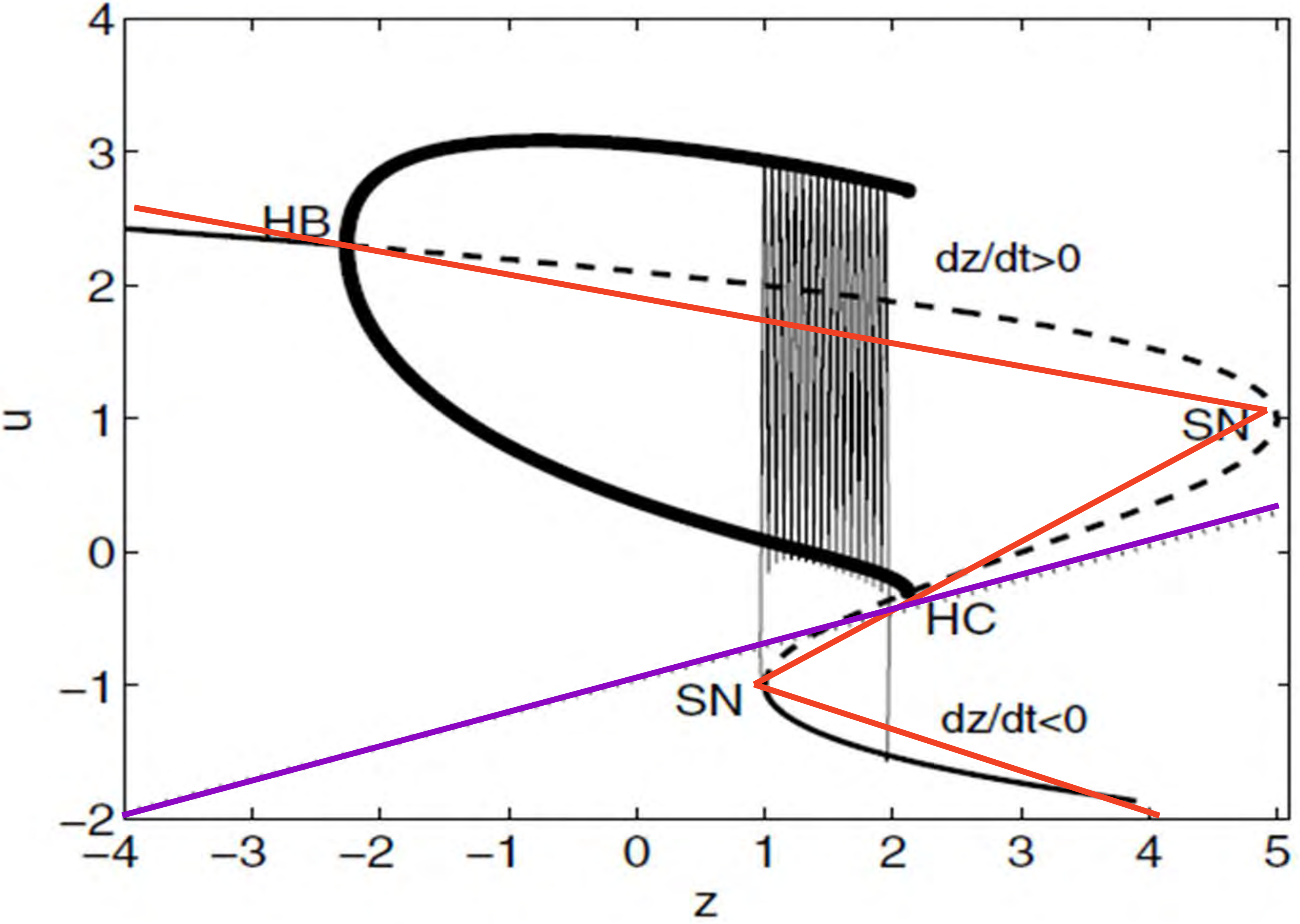
slow →

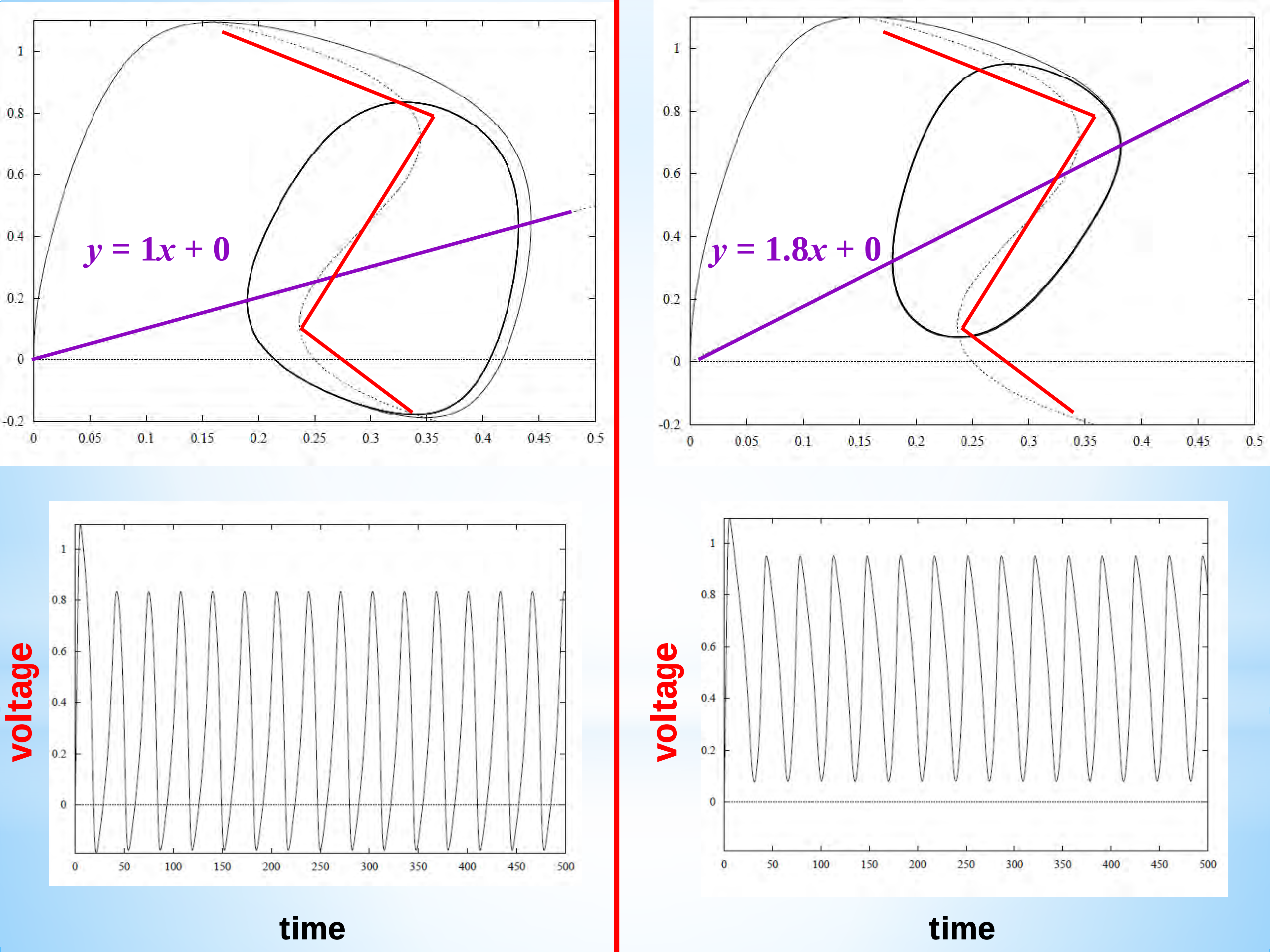
(3) $$\frac{dz}{dt} = \epsilon[h(u) - z]$$

Linear functions **are very important in neuroscience research!!!**

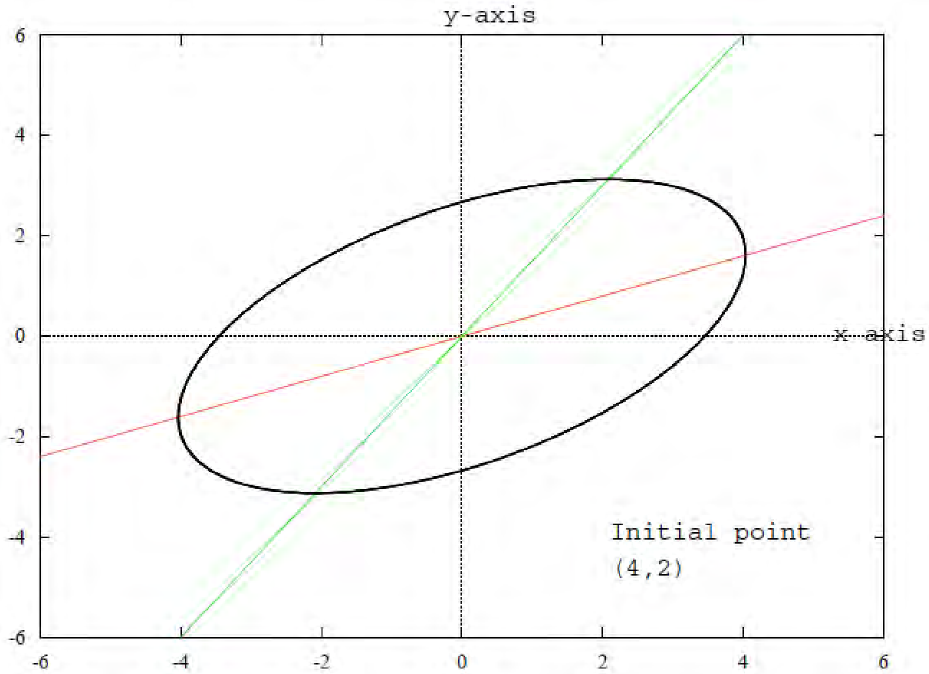
We began to analyze the "equilibrium" of a system like this with a ***linear function*****!**

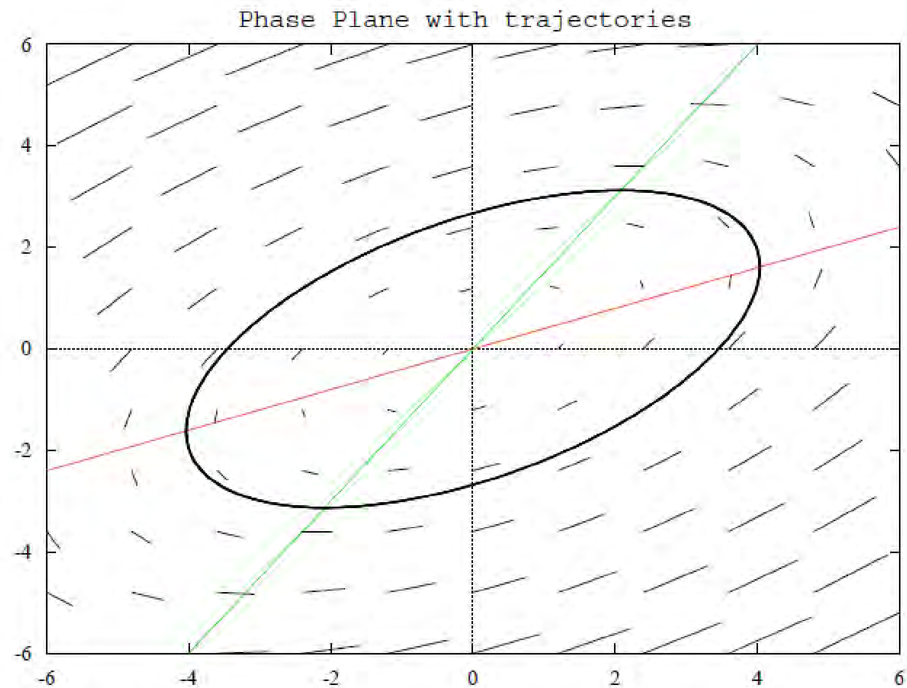
PHASE PLANE ANALYSIS

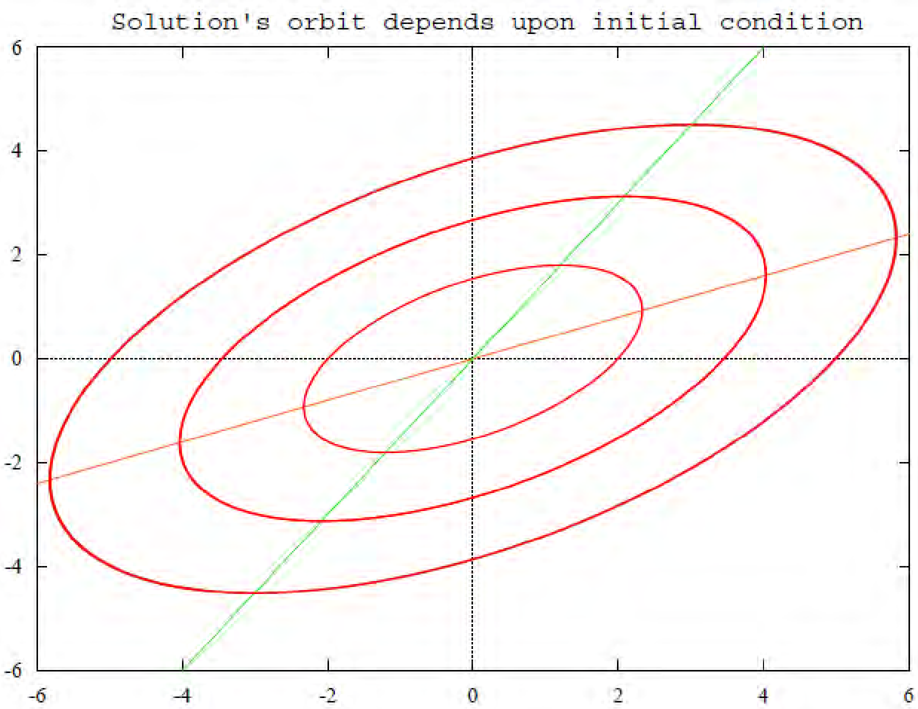


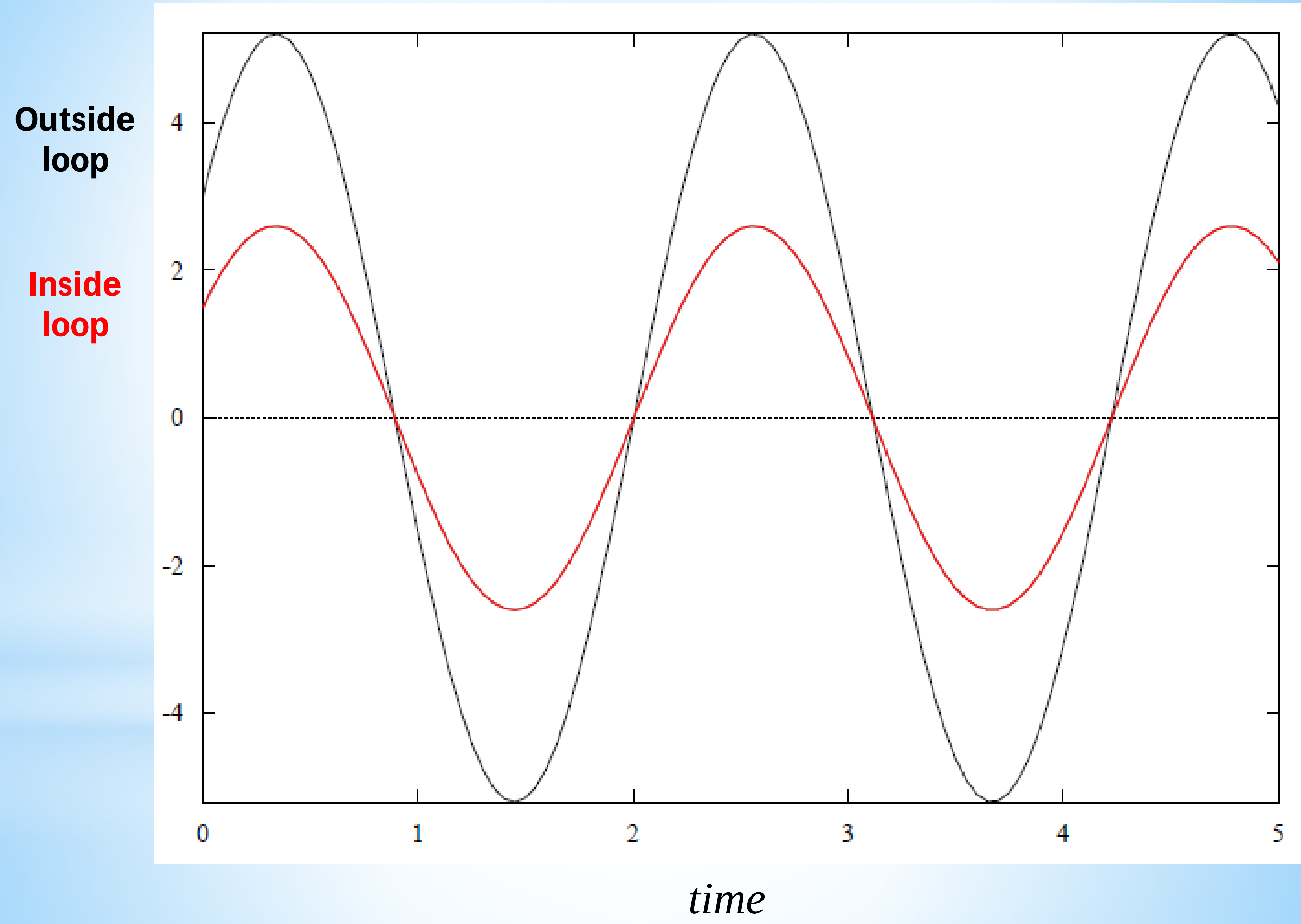


y = 1x + 0
y = 1.8x + 0
voltage
time
voltage
time

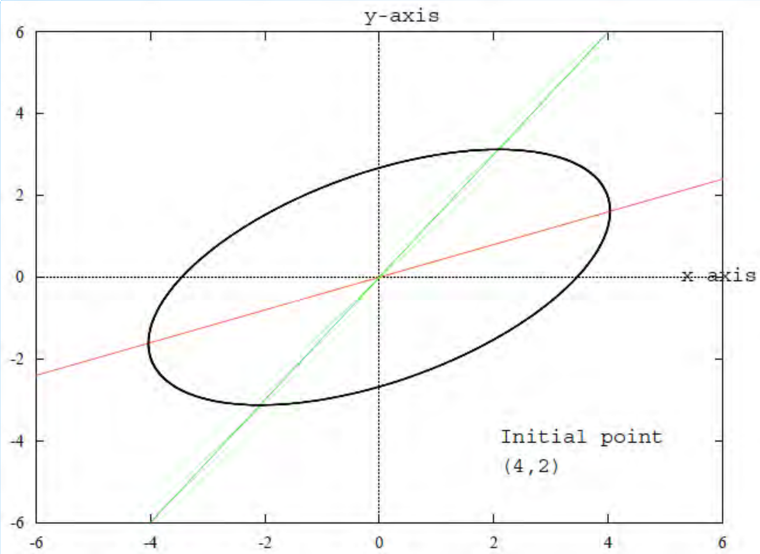

y-axis
x axis
Initial point
(4,2)
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4
6

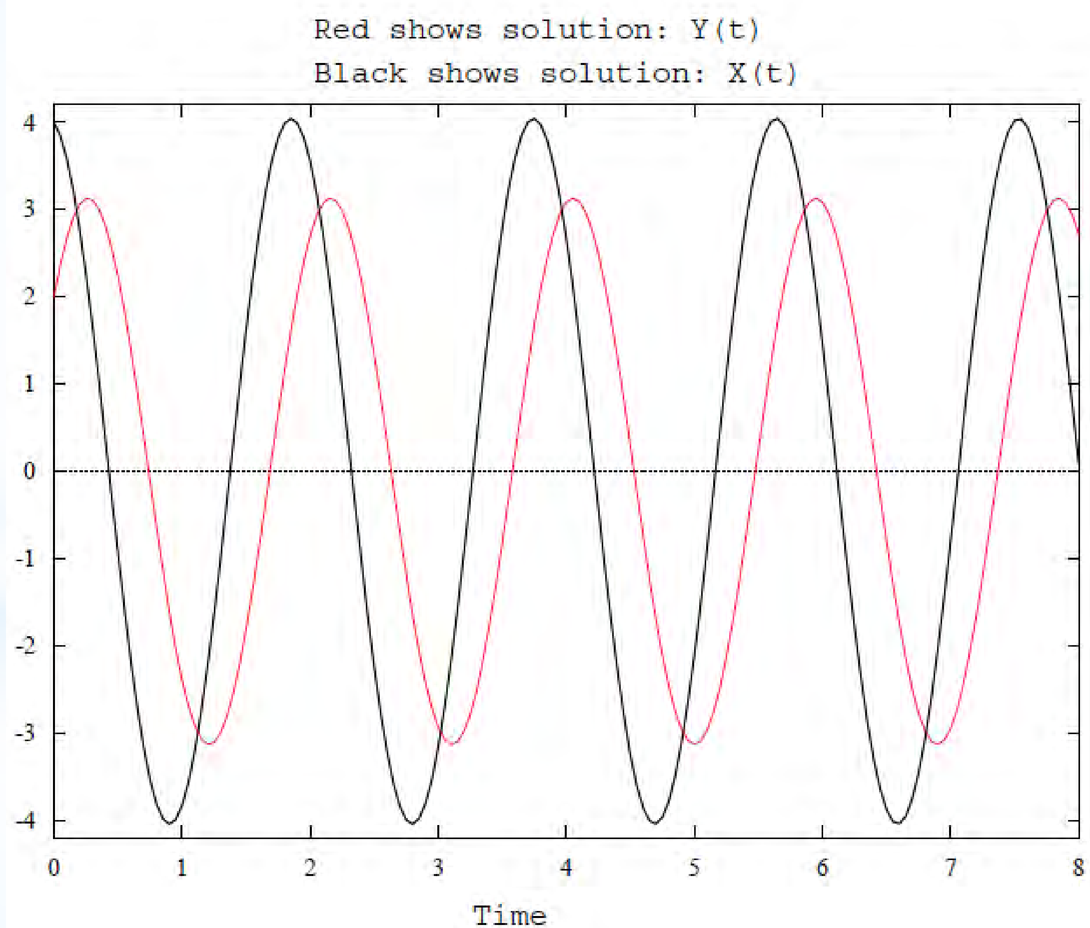

Phase Plane with trajectories
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4
6


Solution's orbit depends upon initial condition
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4
6



Outside loop
Inside loop
4
2
0
-2
-4
0
1
2
3
4
5
time


y-axis
x axis
Initial point
(4,2)


Red shows solution: Y(t)
Black shows solution: X(t)
Time

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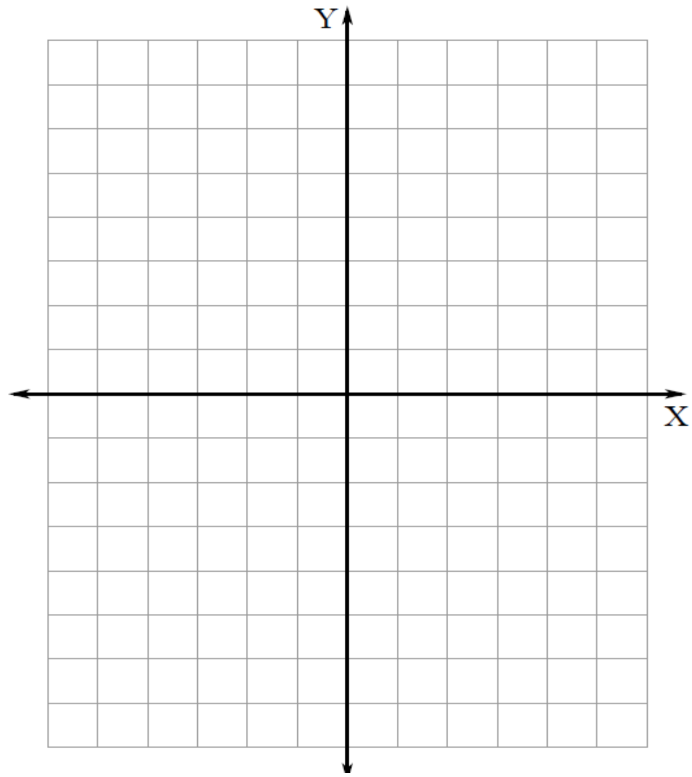
* Izhikevich, Eugene M. ***DYNAMICAL SYSTEMS IN NEUROSCIENCE: THE GEOMETRY OF EXCITABILITY AND BURSTING.*** The MIT Press, 2007.
* Mainen, Zachary F.; Sejnowski, Terrence J. ***RELIABILITY OF SPIKE TIMING IN NEOCORTICAL NEURONS.*** Science, New Series, vol. 268, No. 5216, pp. 1503-1506, 1995.
* Wang, Jiaoyan; Su, Jianzhong; Gonzalez, Humberto Perez; Rubin, Jonathan. ***A RELIABILITY STUDY OF SQUARE WAVE BURSTING β-CELLS WITH NOISE.*** Discrete and Continuous Dynamical systems Series B, Vol. 16, No. 2, 2011.
* Chay, T.R.; Keizer, J. ***MINIMAL MODEL FOR MEMBRANE OSCILLATIONS IN THE PANCREATIC BETA-CELL***. Biophys. J., Vol. 42, pp. 181, 1983.

*REFERENCES

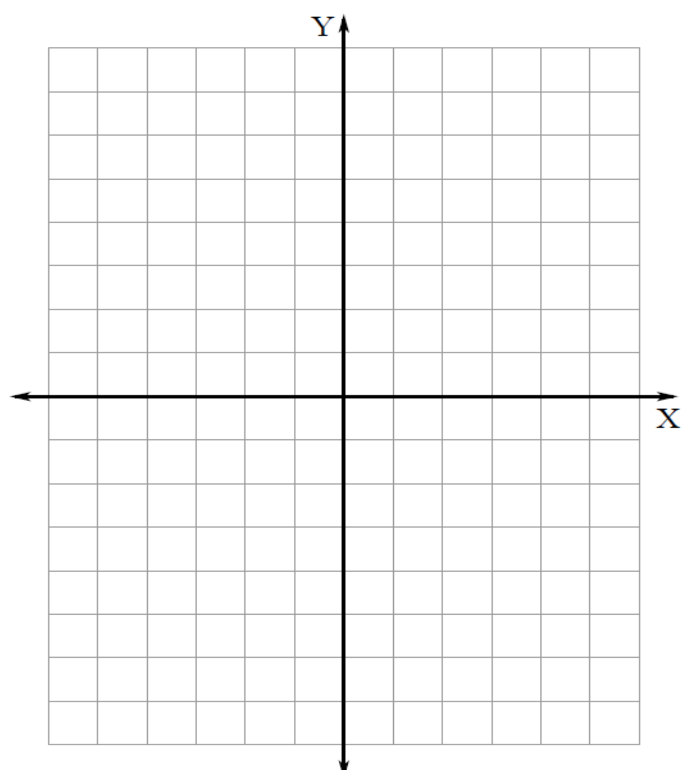
NAME____ PERIOD __________

SYSTEMS OF LINEAR INEQUALITIES

1. Solve and graph $3y < x - 3$ (Shade with colored pencil)



2. Solve and graph $4x + y \geq 0$ and $2x - 3y < 9$
(Shade with different colors)



NAME____ PERIOD _________

SYSTEMS OF LINEAR INEQUALITIES

What steps did you take?

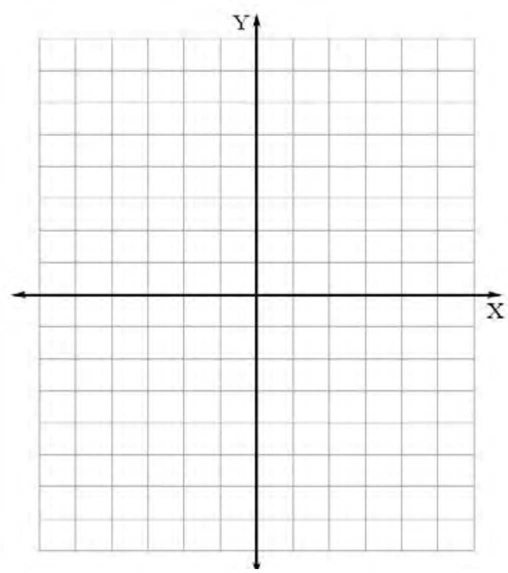
1. Solve for the variable _______ in each _______________________.
2. Graph each line either ________________ or _________________.
3. Shade only ______ side of each line; use different ________________.
4. Determine the _________________ region.

MORE PRACTICE . . .

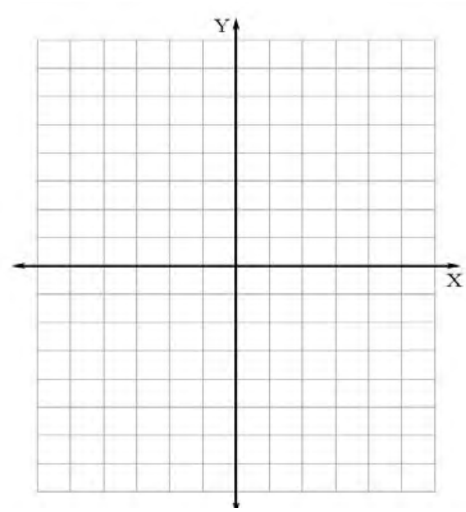
Shade *only* the solution region.

3. $y-6 < -2x-3$ and $5x-2y<10$

   Show why (4,0) is *not* a solution.



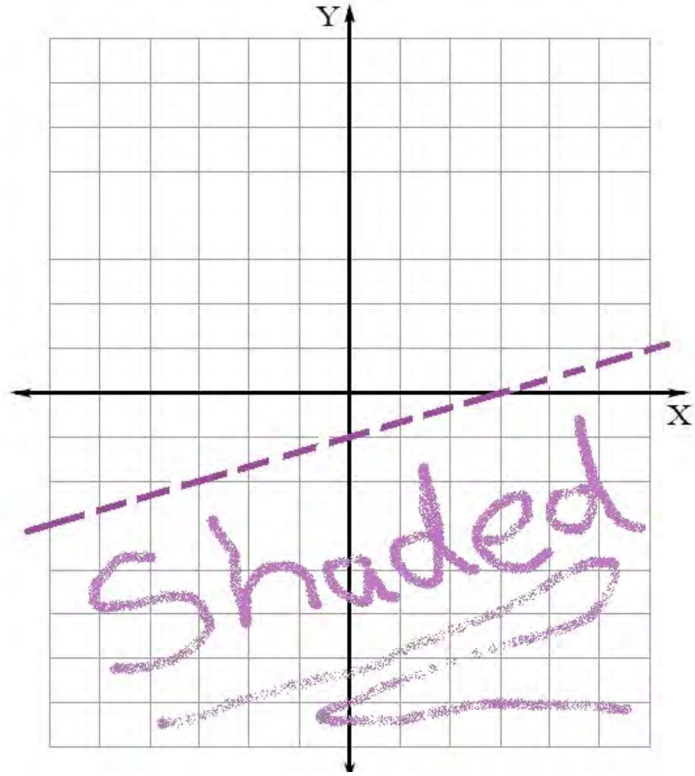
4. $x-y<3$ and $x-y \geq -2$



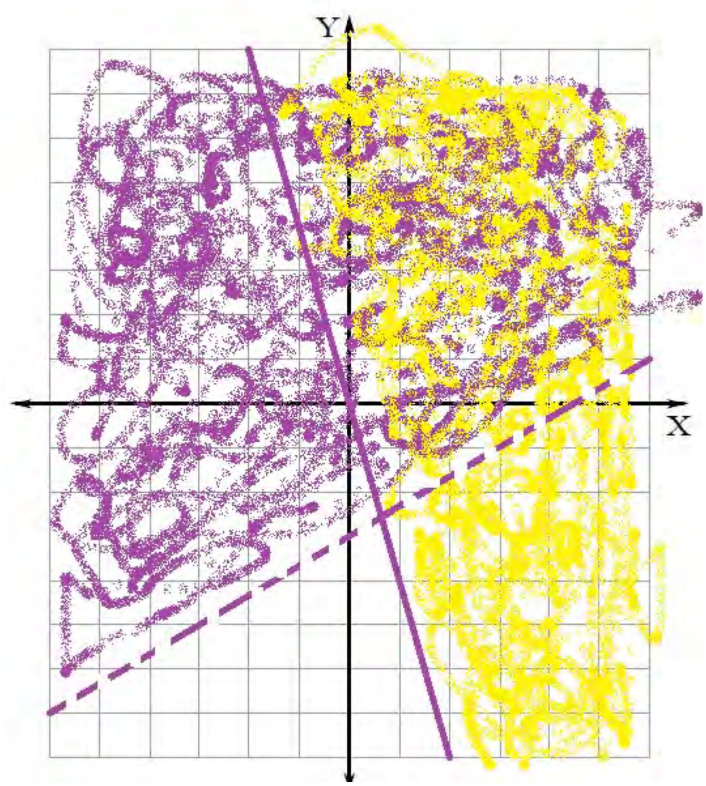
NAME_______________________________________ PERIOD _________

SYSTEMS OF LINEAR INEQUALITIES

1. Solve and graph $3y < x - 3$ (Shade with colored pencil)



2. Solve and graph $4x + y \geq 0$ and $2x - 3y < 9$
(Shade with different colors)



NAME_______________________________________ PERIOD ________

SYSTEMS OF LINEAR INEQUALITIES

What steps did you take?

1. Solve for the variable __y__ in each __inequality__.
2. Graph each line either __solid__ or __dashed__.
3. Shade only __one__ side of each line; use different __colors__.
4. Determine the __solution__ region.

MORE PRACTICE . . .

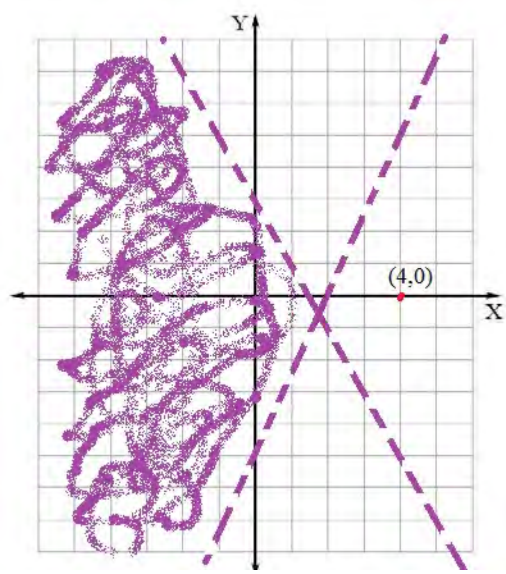
Shade *only* the solution region.

3. $y - 6 < -2x - 3$ and $5x - 2y < 10$

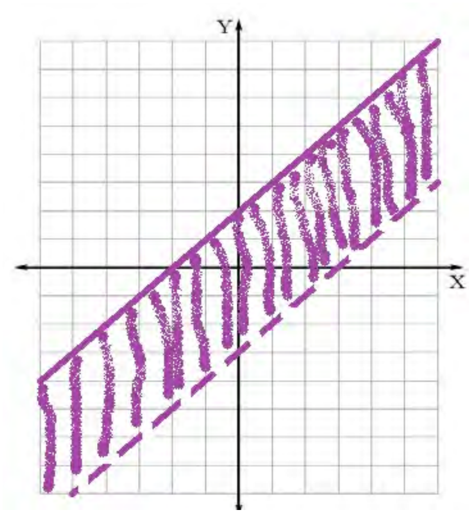
   Show why (4,0) is *not* a solution.

   (0) - 6 < -2(4) - 3

   -6 < -11 is not true



4. $x - y < 3$ and $x - y \geq -2$



GROUP NAMES ____ PERIOD ______

PHASE PLANE ANALYSIS IS A "CAKE WALK"

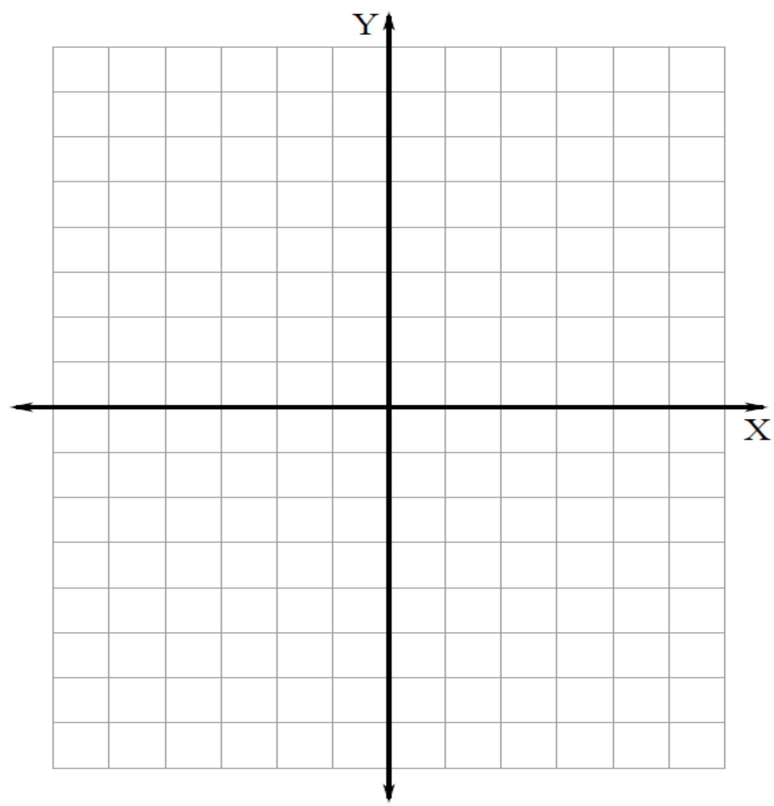
Let's say y models some cell membrane potential (voltage), and x models the free calcium in the cell. Both variables change over time.

Then each equation in the following system describes a ***change*** (in calcium and voltage respectively):

$$\frac{dx}{dt} = 2x - 5y \qquad \text{and} \qquad \frac{dy}{dt} = 3x - 2y$$

STEP ONE Solve the system of linear inequalities assigned to you to earn your first piece of "cake":

$\begin{cases} 2x - 5y < 0 \\ 3x - 2y > 0 \end{cases}$ $\begin{cases} 2x - 5y < 0 \\ 3x - 2y < 0 \end{cases}$ $\begin{cases} 2x - 5y > 0 \\ 3x - 2y < 0 \end{cases}$ $\begin{cases} 2x - 5y > 0 \\ 3x - 2y > 0 \end{cases}$



Of the given points, plot only the ones that lie in your solution-region.

(4 , 2) (3 , 3) (1 , 3) (-1 , 2.2) (-4 , -1)
(-4 , -2) (-3 , -3) (-1 , -3) (1 , – 2.2) (4 , 1)

Your group will determine trajectories for the following point(s):

____

GROUP NAMES ____ PERIOD _______

STEP TWO Find at least one trajectory in your solution-region for another piece of "cake".

Initial point (x,y)	Plug into: $2x - 5y$ $3x - 2y$	Add to initial point
Example: (-1,1)	2(-1) - 5(1) = -7 3(-1) – 2(1) = -5	(-1,1) + (-7,-5) = (-8,-4)

STEP THREE Position trajectories on the floor graph.
(One person per group)

You will be given an arrow for each trajectory you have. The bottom of your arrow should be placed on your initial point. The arrow should point toward your ending point.

STEP FOUR Repeat! Ready for more cake?!

Repeat steps two and three for any point(s) you choose which lie(s) in your solution-region.

OBSERVATIONS & NOTES

1. In STEP ONE, where did the lines intersect? __________________
   a. This is called the ___________________________ point of the system.
   b. The lines are called _________________________.
2. In which direction did trajectories in your solution-region point?
   a. UP-LEFT b. DOWN-LEFT c. DOWN-RIGHT d. UP-RIGHT
3. Based on the behavior of the trajectories, how should we characterize the system's equilibrium?
   a. Unstable: Trajectories moved away from equilibrium
   b. Stable-periodic: Trajectories encircled the equilibrium
   c. Asymptotically stable: Trajectories reached the equilibrium.

GROUP NAMES ____ PERIOD ______

PHASE PLANE ANALYSIS IS A "CAKE WALK"

Let's say y models some cell membrane potential (voltage), and x models the free calcium in the cell. Both variables change over time.

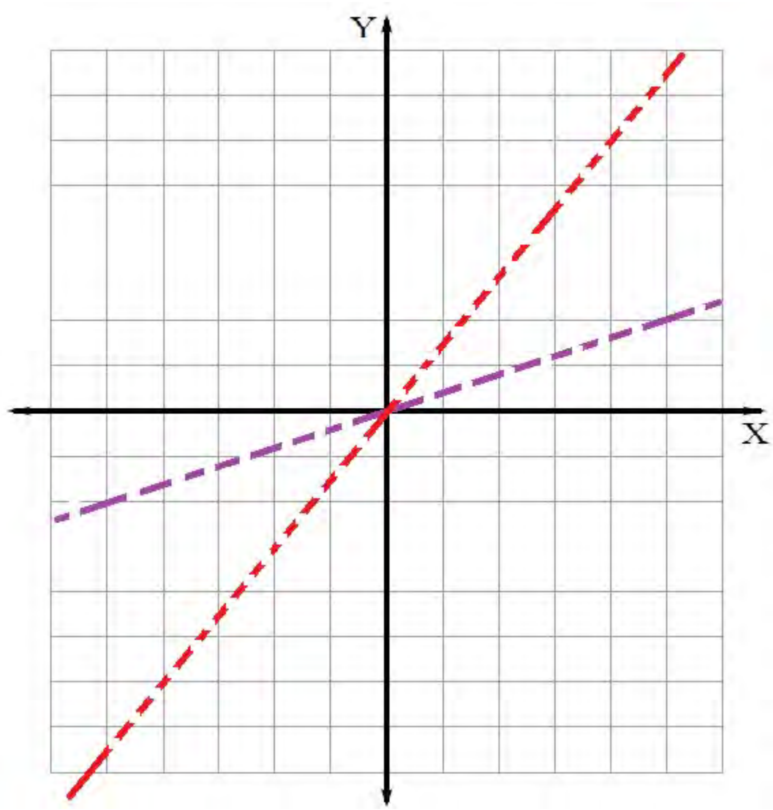
Then each equation in the following system describes a ***change*** (in calcium and voltage respectively):

$$\frac{dx}{dt} = 2x - 5y \qquad \text{and} \qquad \frac{dy}{dt} = 3x - 2y$$

STEP ONE Solve the system of linear inequalities assigned to you to earn your first piece of "cake":

$\begin{cases} 2x - 5y < 0 \\ 3x - 2y > 0 \end{cases}$ $\begin{cases} 2x - 5y < 0 \\ 3x - 2y < 0 \end{cases}$ $\begin{cases} 2x - 5y > 0 \\ 3x - 2y < 0 \end{cases}$ $\begin{cases} 2x - 5y > 0 \\ 3x - 2y > 0 \end{cases}$

Each group is assigned one of the four systems above. So the shading (solution region) will be different for each group.



Of the given points, plot only the ones that lie in your solution-region.

(4 , 2) (3 , 3) (1 , 3) (-1 , 2.2) (-4 , -1)

(-4 , -2) (-3 , -3) (-1 , -3) (1 , – 2.2) (4 , 1)

Your group will determine trajectories for the following point(s):

See underlined points for each group.

GROUP NAMES ____ PERIOD _______

STEP TWO Find at least one trajectory in your solution-region for another piece of "cake".

half of the points are shown below:

Initial point (*x*,*y*)	**Plug into: 2*x* – 5*y* 3*x* – 2*y***	**Add to initial point**
Example: (-1,1)	2(-1) - 5(1) = -7 3(-1) – 2(1) = -5	(-1,1) + (-7,-5) = (-8,-4)
(4,2)	2(4) - 5(2) = -2 3(4) - 2(2) = 8	(4,2) + (-2,8) = (2,10)
(3,3)	2(3) - 5(3) = -9 3(3) - 2(3) = 3	(3,3) + (-9,3) = (-6,6)
(1,3)	2(1) - 5(3) = -13 3(1) - 2(3) = -3	(1,3) + (-13,-3) = (-12,0)
(-1,2.2)	2(-1) - 5(2.2) = -13 3(-1) - 2(2.2) = -7.4	(-1,2.2) + (-13,-7.4) = (-14,-5.2)
(-4,-1)	2(-4) - 5(-1) = -3 3(-4) - 2(-1) = -10	(-4,-1) + (-3,-10) = (-7,-11)

STEP THREE Position trajectories on the floor graph.
(One person per group)

You will be given an arrow for each trajectory you have. The bottom of your arrow should be placed on your initial point. The arrow should point toward your ending point.

STEP FOUR Repeat! Ready for more cake?!

Repeat steps two and three for any point(s) you choose which lie(s) in your solution-region.

OBSERVATIONS & NOTES

1. In STEP ONE, where did the lines intersect? origin
   a. This is called the equilibrium point of the system.
   b. The lines are called nullclines.
2. In which direction did trajectories in your solution-region point?
   a. UP-LEFT b. DOWN-LEFT c. DOWN-RIGHT d. UP-RIGHT different for each group
3. Based on the behavior of the trajectories, how should we characterize the system's equilibrium?
   a. Unstable: Trajectories moved away from equilibrium
   b. Stable-periodic: Trajectories encircled the equilibrium
   c. Asymptotically stable: Trajectories reached the equilibrium.

PHASE PLANE KEY

Initial point	Calculated trajectory	Points toward	Arrow color
Group 1			
(4,2)	(-2,8)	**(2,10)**	**GREEN**
(3,3)	(-9,3)	**(-6,6)**	**PINK**
Group 2			
(1,3)	(-13,-3)	**(-12,0)**	**RED**
(-1,2.2)	(-13,-7.4)	**(-14,-5.2)**	**RED**
(-4,-1)	(-3,-10)	**(-7,-11)**	**GREEN**
Group 3			
(-4,-2)	(2,-8)	**(-2,-10)**	**GREEN**
(-3,-3)	(9,-3)	**(6,-6)**	**PINK**
Group 4			
(-1,-3)	(13,3)	**(12,0)**	**RED**
(1,-2.2)	(13,7.4)	**(14,5.2)**	**RED**
(4,1)	(3,10)	**(7,11)**	**GREEN**

IF YOU HAVE MORE THAN 4 GROUPS, THEN MORE THAN ONE GROUP CAN BE ASSIGNED THE SAME INITIAL POINTS.